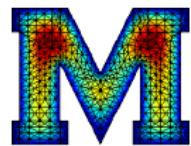


NUMERICAL METHODS FOR FRACTIONAL DIFFUSION

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Outline

Motivation

The Integral Laplacian

The Spectral Laplacian

Dunford-Taylor Approach

Extensions

Open Problems

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Open Problems

Local Jump Random Walk

- Consider a random walk of a particle along the real line.
- $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$ — possible states of the particle.
- $u(x, t)$ — probability of the particle to be at $x \in h\mathbb{Z}$ at time $t \in \tau\mathbb{N}$.
- Local jump random walk:** at each time step of size τ , the particle jumps to the left or right with probability $1/2$.



$$u(x, t + \tau) = \frac{1}{2}u(x + h, t) + \frac{1}{2}u(x - h, t)$$

If we consider $2\tau = h^2$, then we obtain

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{u(x + h, t) + u(x - h, t) - 2u(x, t)}{h^2}$$

Letting $h, \tau \downarrow 0$ yields the **heat equation**

$$u_t - \Delta u = 0$$

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Long Jump Random Walk

- The probability that the particle jumps from the point $hk \in h\mathbb{Z}$ to the point $hm \in h\mathbb{Z}$ is $\mathcal{K}(k - m) = \mathcal{K}(m - k)$:



$$u(x, t + \tau) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) u(x + hk, t).$$

- No-time memory:** Since $\sum_{k \in \mathbb{Z}} \mathcal{K}(k) = 1$, this yields

$$u(x, t + \tau) - u(x, t) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) (u(x + hk, t) - u(x, t))$$

- If $\mathcal{K}(y) \sim |y|^{-(1+2s)}$ with $s \in (0, 1)$ and $\tau = h^{2s}$, then $\frac{\mathcal{K}(k)}{\tau} = h\mathcal{K}(kh)$. Letting $h, \tau \downarrow 0$ yields the **fractional heat equation**

$$\partial_t u = \int_{\mathbb{R}} \frac{u(x+y, t) - u(x, t)}{|y|^{1+2s}} dy \quad \Leftrightarrow \quad \partial_t u + (-\Delta)^s u = 0.$$

- Long-range time memory:** $\partial_t u \Rightarrow \partial_t^\gamma u$ ($0 < \gamma < 1$)

$$\partial_t^\gamma u + (-\Delta)^s u = 0.$$

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Applications of Nonlocal Operators and Fractional Diffusion

- ▶ Modeling anomalous diffusion (Metzler, Klafter 2000, 2004).
- ▶ Peridynamics (Silling 2000; Du, Gunzburger 2012; Lipton 2015).
- ▶ Modeling contaminant transport in porous media (Benson et al 2000; Seymour et al 2007).
- ▶ Finance (Carr et al. 2002; Matache, Schwab, von Petersdorff et al. 2004).
- ▶ Lévy processes (Bertoin 1996; Farkas, Reich, Schwab 2007).
- ▶ Nonlocal field theories (Eringen 1972, 2002).
- ▶ Materials science (Bates 2006).
- ▶ Image processing (Gilboa, Osher 2008).
Caffarelli-Silvestre extension → (Gatto, Hesthaven 2014)
Spectral method → (Bartels, Antil 2017).
- ▶ Fractional Navier Stokes equation (Li et al 2012; Debbi 2014):

$$u_t + u \cdot \nabla u + (-\Delta)^s u + \nabla p = 0$$

- ▶ Quasi-geostrophic equation (Karniadakis 2017)
- ▶ Fractional Cahn Hilliard equation (Segatti, 2014; Ainsworth 2017)

The domain Ω can be quite general!

Nonlocal Models: Historical Remarks

- **Nonlocal continuum physics:**

- ▶ A.C. Eringen and D.G.B. Edelen, *On nonlocal elasticity*, International Journal of Engineering Science, 10 (1972), 233-248 (1427 google scholar citations).
- ▶ A.C. Eringen, *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, J. Appl. Phys. 54, 4703 (1983). (2354 google scholar citations).
- ▶ A.C. Eringen, **Nonlocal Continuum Field Theories**, Springer (2002).

Nonlocal continuum field theories are concerned with material bodies whose behavior at any interior point depends on the state of all other points in the body – rather than only on an effective field resulting from these points – in addition to its own state and the state of some calculable external field.

- Recent developments:

- ▶ Peridynamics: S.A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, Journal of the Mechanics and Physics of Solids (2000) (1322 google scholar citations).
- ▶ Dirichlet-to-Neumann map: L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Communications in Partial Differential Equations, (2007) (1392 google scholar citations).

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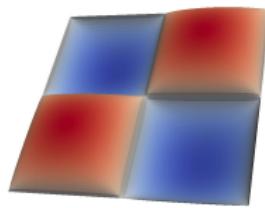
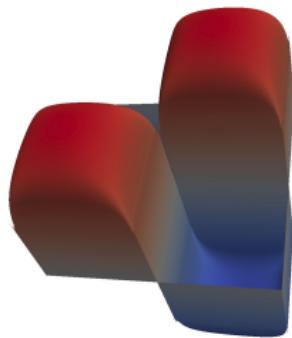
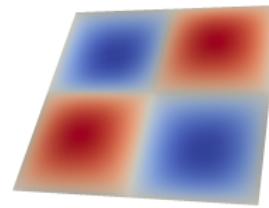
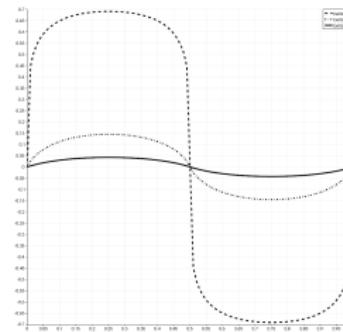
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$(-\Delta u)^s = f$: Varying s for Discontinuous Checkerboard f  $s = 0.5$  $s = 0.1$  $s = 0.8$ cut at $y = 0.25$

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Nonlocal Operator: Definition in \mathbb{R}^d for $d \geq 1$

Let $s \in (0, 1)$ and $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be smooth enough (belongs to Schwartz class \mathcal{S}).

- Fourier transform:

$$\mathcal{F}((-Δ)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)$$

- Integral representation:

$$(-Δ)^s u(x) = C(d, s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(x')}{|x - x'|^{d+2s}} dx',$$

where $C(d, s) = \frac{2^{2s} s \Gamma(s + \frac{d}{2})}{\pi^{d/2} \Gamma(1-s)}$ is a normalization constant involving the Gamma-function Γ .

- Pointwise limits $s \rightarrow 0, 1$: there holds

$$\lim_{s \rightarrow 0} (-Δ)^s u = u,$$

$$\lim_{s \rightarrow 1} (-Δ)^s u = -Δu.$$

Nonlocal Operator: Integral Definition for Bounded Domain $\Omega \subset \mathbb{R}^d$

Let $\Omega \subset \mathbb{R}^d$ be open, with smooth boundary, and let $f : \Omega \rightarrow \mathbb{R}$ be smooth.

- **Boundary value problem:**

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c = \mathbb{R}^d \setminus \Omega. \end{cases}$$

- **Integral representation:**

$$(-\Delta)^s u(x) = C(d, s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(x')}{|x - x'|^{d+2s}} dx' = f(x) \quad x \in \Omega.$$

- **Probabilistic interpretation:** It is the same as over \mathbb{R}^d except that particles are killed upon reaching Ω^c .

Function Spaces

- **Fractional Sobolev space in \mathbb{R}^d :**

$$H^s(\mathbb{R}^d) = \left\{ w \in L^2(\mathbb{R}^d) : |w|_{H^s(\mathbb{R}^d)} < \infty \right\}$$

with

$$\langle u, w \rangle := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(x'))(w(x) - w(x'))}{|x - x'|^{d+2s}} dx' dx,$$

$$|w|_{H^s(\mathbb{R}^d)} := \langle w, w \rangle^{\frac{1}{2}}, \quad \|w\|_{H^s(\mathbb{R}^d)} := \left(|w|_{H^s(\mathbb{R}^d)}^2 + \|w\|_{L^2(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}.$$

- **Fractional Sobolev space in Ω :**

$$\mathbb{H}^s(\Omega) := \left\{ w|_\Omega : w \in H^s(\mathbb{R}^d), w|_{\mathbb{R}^d \setminus \Omega} = 0 \right\} = \begin{cases} H^s(\Omega) & s \in (0, \frac{1}{2}) \\ H_{00}^{\frac{1}{2}}(\Omega) & s = \frac{1}{2} \\ H_0^s(\Omega) & s \in (\frac{1}{2}, 1). \end{cases}$$

Equivalent norms: using Poincaré inequality in $\mathbb{H}^s(\Omega)$

$$\|w\|_{\mathbb{H}^s(\Omega)} := \|w\|_{H^s(\mathbb{R}^d)} \approx \begin{cases} \|w\|_{H^s(\Omega)} & 0 < s < \frac{1}{2} \\ |w|_{H^s(\Omega)} & \frac{1}{2} < s < 1 \end{cases} \quad \forall w \in \mathbb{H}^s(\Omega).$$

Variational Formulation

- **Bilinear form in $\mathbb{H}^s(\Omega)$:**

$$\llbracket u, w \rrbracket := \frac{C(d, s)}{2} \underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(x'))(w(x) - w(x'))}{|x - x'|^{d+2s}} dx' dx}_{= \langle u, w \rangle}$$

This form is symmetric, continuous and coercive, and equivalent to the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{H}^s(\Omega)$; recall **Poincaré inequality**

$$\|w\|_{L^2(\Omega)} \leq c(\Omega, n, s) |w|_{H^s(\mathbb{R}^d)} \quad \forall w \in \mathbb{H}^s(\Omega).$$

- **Variational formulation:** for any $f \in \mathbb{H}^{-s}(\Omega) = \text{dual of } \mathbb{H}^s(\Omega)$, consider

$$u \in \mathbb{H}^s(\Omega) : \quad \llbracket u, w \rrbracket = (f, w) \quad \forall w \in H^s(\Omega),$$

where (\cdot, \cdot) stands for the duality pairing. Existence, uniqueness, and stability follows from Lax-Milgram.

Boundary Behavior: Sobolev Regularity of Solutions (Grubb (2015))

- **Theorem** (Vishik & Eskin (1965), Grubb (2015)). If $f \in H^r(\Omega)$ for some $r \geq 0$ and $\partial\Omega \in C^\infty$, then for all $\varepsilon > 0$

$$u \in \begin{cases} H^{2s+r}(\Omega) & \text{if } s + r < 1/2, \\ H^{s+1/2-\varepsilon}(\Omega) & \text{if } s + r \geq 1/2. \end{cases}$$

The Dirichlet boundary conditions preclude further gain of regularity beyond
 $H^{s+1/2-\varepsilon}(\Omega)$.

- **Example:** If $\Omega = B(0, r)$ and $f \equiv 1$, then the solution u is given by

$$u(x) = C(r^2 - |x|^2)_+^s,$$

which does not belong to $H^{s+1/2}(\Omega)$. The regularity above is sharp!

- **Boundary behavior** (Grubb (2015)). If $\partial\Omega \in C^\infty$ then

$$u(x) \approx \text{dist}(x, \partial\Omega)^s + v(x)$$

with v smooth. Singular boundary behavior regardless of smoothness of $\partial\Omega$.

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Weighted Fractional Sobolev Regularity (Acosta & Borthagaray (2017))

- **Definition of space $H_\alpha^{1+\theta}(\Omega)$:** Let $\alpha \geq 0$ and $\theta \in (0, 1)$.

$$\|v\|_{H_\alpha^{1+\theta}(\Omega)}^2 := \|v\|_{H^1(\Omega)}^2 + \iint_{\Omega \times \Omega} \frac{|Dv(x) - Dv(y)|^2}{|x - y|^{n+2\theta}} \delta(x, y)^{2\alpha} dx dy$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$ and $\delta(x, x') = \min\{\delta(x), \delta(x')\}$.

- **Weighted estimates:** Let $0 < s < 1$, $f \in C^{1-s}(\Omega)$, and $\varepsilon > 0$ small. Then, the solution u of $(-\Delta)^s u = f$ which vanishes in Ω^c belongs to $H_{1/2-\varepsilon}^{1+s-2\varepsilon}(\Omega)$ and satisfies the estimate

$$\|u\|_{H_{1/2-\varepsilon}^{1+s-2\varepsilon}(\Omega)} \leq \frac{C(\Omega, s)}{\varepsilon} \|f\|_{C^{1-s}(\Omega)}.$$

(This is based on results by Ros-Oton and Serra (2014)).

FEM and Best Approximation

- **Mesh:** Let \mathcal{T} be a shape-regular and quasi-uniform mesh of Ω of size h .
- **Finite element space:** Let

$$\mathbb{U}(\mathcal{T}) = \{v \in C^0(\overline{\Omega}): v|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}\}.$$

- **Discrete problem:** Find $U \in \mathbb{U}(\mathcal{T})$ such that

$$[U, W] = (f, W) \quad \forall W \in \mathbb{U}(\mathcal{T}).$$

- **Best approximation:** Since we project over $\mathbb{U}(\mathcal{T})$ with respect to the energy norm $|\cdot|_{\mathbb{H}^s(\Omega)}$ induced by $[\cdot, \cdot]$, we get

$$|u - U|_{\mathbb{H}^s(\Omega)} = \min_{W \in \mathbb{U}(\mathcal{T})} |u - W|_{\mathbb{H}^s(\Omega)}.$$

- **A priori error analysis:** must account for nonlocality and boundary behavior.

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A Priori Error Analysis: Interpolation estimates in $\mathbb{H}^s(\Omega)$

- Localized estimates in $H^s(\Omega)$ (Faermann (2002)):

$$|w|_{H^s(\Omega)}^2 \leq \sum_{K \in \mathcal{T}} \left[\int_K \int_{S_K} \frac{|w(x) - w(x')|^2}{|x - x'|^{d+2s}} dx' dx + \frac{C(d, \sigma)}{sh_K^{2s}} \|w\|_{L^2(K)}^2 \right],$$

where S_K is the patch associated with $K \in \mathcal{T}$ and σ is the shape regularity constant of \mathcal{T} .

- Error estimates for quasi-uniform meshes (Acosta-Borthagaray (2017))

$$|u - U|_{\mathbb{H}^s(\Omega)} \leq C(s, \sigma) h^{\frac{1}{2}} |\ln h| \|f\|_{H^{1/2-s}(\Omega)}.$$

- Example: $u(x) = C(r^2 - |x|^2)_+^s$ with $\Omega = B(0, 1) \subset \mathbb{R}^2$, $f = 1$

s	0.1	0.3	0.5	0.7	0.9
Order	0.497	0.498	0.501	0.504	0.532

Rate is quasi-optimal! Q: Is it possible to improve the order of convergence?

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Error Estimates in Graded Meshes (Acosta & Borthagaray (2017))

- **Weighted quasi-interpolation:**

$$\int_T \int_{S_T} \frac{|(v - \Pi_h v)(x) - (v - \Pi_h v)(x')|^2}{|x - x'|^{n+2s}} dx' dx \leq Ch_T^{2(1+\theta-\alpha-s)} |v|_{H_\alpha^{1+\theta}(S_T)}^2.$$

- **Energy error estimate:** Let $d = 2$ and \mathcal{T} be a graded mesh satisfying

$$h_K \leq C(\sigma) \begin{cases} h^2, & K \cap \partial\Omega \neq \emptyset, \\ h \operatorname{dist}(K, \partial\Omega)^{1/2}, & K \cap \partial\Omega = \emptyset, \end{cases}$$

whence $\#\mathcal{T} \approx h^{-2} |\log h|$. If $0 < s < 1$, then

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim (\#\mathcal{T})^{-\frac{1}{2}} |\log(\#\mathcal{T})| \|f\|_{C^{1-s}(\bar{\Omega})}.$$

- **Improvement:** This also reads $\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h |\log h| \|f\|_{C^{1-s}(\bar{\Omega})}$ and is thus **first order**.

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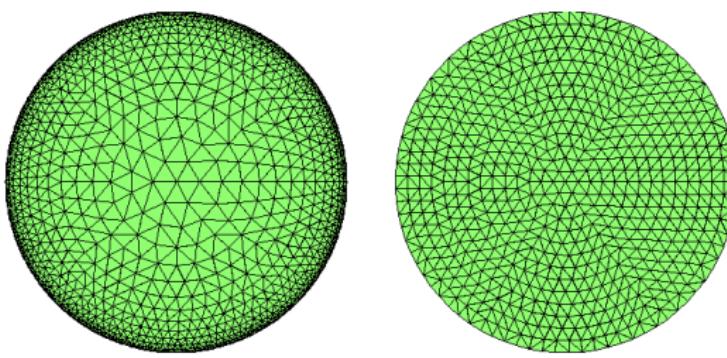
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Numerical Experiment (Acosta & Borthagaray (2017))

Exact solution: $u(x) = C(r^2 - |x|^2)_+^s$ with $\Omega = B(0, 1) \subset \mathbb{R}^2$, $f = 1$.

Experiment with either uniform or graded \mathcal{T} : let $h_K \approx h \operatorname{dist}(K, \partial\Omega)^{1/2}$

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Uniform \mathcal{T}	0.497	0.496	0.498	0.500	0.501	0.505	0.504	0.503	0.532
Graded \mathcal{T}	1.066	1.040	1.019	1.002	1.066	1.051	0.990	0.985	0.977



Optimality: First order accuracy $\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h |\log h|$ seems optimal.

Implementation in 2d (Acosta, Bersetche & Borthagaray (2017))

- **Basis functions:** $\{\phi_i\}_{i=1}^I \Rightarrow \text{span } \{\phi_i\}_{i=1}^I = \mathbb{U}(\mathcal{T})$.
- **Matrix formulation:** If $A = (a_{ij})_{ij=1}^I$ and $Q = (\Omega \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \Omega)$ with

$$a_{i,j} = [\phi_i, \phi_j] = \frac{C(d, s)}{2} \iint_Q \frac{(\phi_i(x) - \phi_i(x'))(\phi_j(x) - \phi_j(x'))}{|x - x'|^{2+2s}} dx' dx.$$

and $\mathbf{U} = (U_i)_{i=1}^I$, $\mathbf{F} = (\langle f, \phi_i \rangle)_{i=1}^I$ satisfy $U = \sum_{i=1}^I U_i \phi_i \in \mathbb{U}(\mathcal{T})$, then

$$A\mathbf{U} = \mathbf{F}.$$

- **Computation:** We have $a_{i,j} = \frac{C(d, s)}{2} \sum_{\ell=1}^I \left(\sum_{m=1}^I I_{\ell,m}^{i,j} + 2J_{\ell}^{i,j} \right)$ with

$$I_{\ell,m}^{i,j} := \int_{K_\ell} \int_{K_m} \frac{(\phi_i(x) - \phi_i(x'))(\phi_j(x) - \phi_j(x'))}{|x - x'|^{2+2s}} dx' dx,$$

$$J_{\ell}^{i,j} := \int_{K_\ell} \int_{B^c} \frac{\phi_i(x)\phi_j(x)}{|x - x'|^{2+2s}} dx' dx.$$

- **Computational difficulties**

- ▶ Non-integrable singularities (use techniques from BEM, Schwab-Sauter (2004))
- ▶ Unbounded domains (improvements by Ainsworth-Glusa (2017))

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Open Problems

Basic Spectral Theory

- **Operator:** $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is symmetric, closed and unbounded and its inverse is compact.
- **Spectral decomposition:** The eigenpairs $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$ satisfy $\lambda_k \geq \lambda_0 > 0$

$$-\Delta \varphi_k = \lambda_k \varphi_k, \quad \varphi_k|_{\partial\Omega} = 0,$$

and $\{\varphi_k\}_{k=1}^{\infty}$ form an orthonormal basis of $L^2(\Omega)$ and orthogonal basis of $H_0^1(\Omega)$.

- **Fractional Laplacian:** For u sufficiently smooth and $0 < s \leq 1$

$$u = \sum_{k=1}^{\infty} u_k \varphi_k \quad \longmapsto \quad (-\Delta)^s u := \sum_{k=1}^{\infty} u_k \lambda_k^s \varphi_k$$

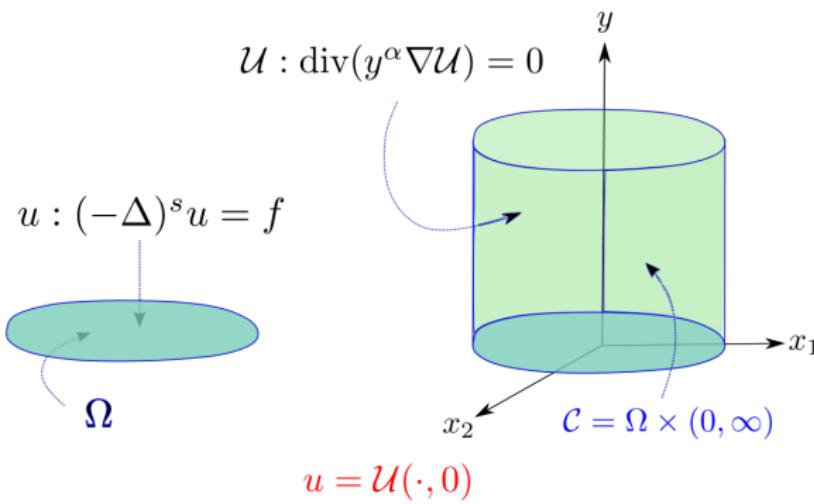
- **Function spaces:** $(-\Delta)^s : \mathbb{H}^s(\Omega) \rightarrow \mathbb{H}^{-s}(\Omega)$, where

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k \varphi_k : \sum_{k=1}^{\infty} \lambda_k^s w_k^2 < \infty \right\} = \begin{cases} H^s(\Omega) & s \in (0, \frac{1}{2}) \\ H_{00}^{\frac{1}{2}}(\Omega) & s = \frac{1}{2} \\ H_0^s(\Omega) & s \in (\frac{1}{2}, 1). \end{cases}$$

The Dirichlet-to-Neumann Map: The Extension Problem for $0 < s < 1$

- **Extension problem:**

- ▶ $\Omega = \mathbb{R}^d$: Caffarelli, Silvestre (2007);
- ▶ $\Omega \subset \mathbb{R}^d$ bounded and $\mathcal{U} = 0$ on $\partial_L \mathcal{C}$: Stinga, Torrea (2010–2012), Cabré, Tan (2010), Capella et al (2011)



- **Parameters:** $s \in (0, 1)$ and $\alpha = 1 - 2s \in (-1, 1)$.
- **Neumann condition:** $\partial_{\nu^\alpha} \mathcal{U} = -\lim_{y \downarrow 0} y^\alpha \partial_y \mathcal{U} = d_s f$ on $\Omega \times \{0\}$.
- **Scaling constant:** $d_s = 2^\alpha \Gamma(1-s)/\Gamma(s)$.

Weak Formulation and Muckenhoupt Weights

- **Space:** $\overset{\circ}{H}_L^1(y^\alpha, \mathcal{C}) = \{w \in L^2(y^\alpha, \mathcal{C}) : \nabla w \in L^2(y^\alpha, \mathcal{C}), w|_{\partial_L \mathcal{C}} = 0\}.$
- **Weak formulation:** seek $\mathcal{U} \in \overset{\circ}{H}_L^1(y^\alpha, \mathcal{C})$ such that

$$\int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla \phi = d_s \langle f, \text{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega), \mathbb{H}^s(\Omega)}, \quad \forall \phi \in \overset{\circ}{H}_L^1(y^\alpha, \mathcal{C}).$$

- **Muckenhoupt class A_2 :** There is a constant C such that for every $a, b \in \mathbb{R}$, with $a > b$,

$$\frac{1}{b-a} \int_a^b |y|^\alpha dy \cdot \frac{1}{b-a} \int_a^b |y|^{-\alpha} dy \leq C.$$

- **Important consequences:**

- ▶ Singular integral operators are continuous on $L^2(y^\alpha, \mathcal{C})$.
- ▶ $H^1(y^\alpha, \mathcal{C})$ is Hilbert and $\mathcal{C}_b^\infty(\mathcal{C})$ is dense.
- ▶ Traces on $\partial_L \mathcal{C}$ are well defined.

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Weighted Sobolev Spaces

- **Weighted Poincaré inequality:** There is a constant C , s.t.

$$\int_{\mathcal{C}} y^\alpha |w|^2 \leq C \int_{\mathcal{C}} y^\alpha |\nabla w|^2 \quad \forall w \in \overset{\circ}{H}_L^1(y^\alpha, \mathcal{C}).$$

- **Surjective trace operator:** $\text{tr}_\Omega : \overset{\circ}{H}_L^1(y^\alpha, \mathcal{C}) \rightarrow \mathbb{H}^s(\Omega)$.
- **Existence and uniqueness:** Lax-Milgram applies for every $f \in \mathbb{H}^{-s}(\Omega)$. Also

$$\|\mathcal{U}\|_{\overset{\circ}{H}_L^1(y^\alpha, \mathcal{C})} = \|u\|_{\mathbb{H}^s(\Omega)} = \sqrt{d_s} \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

- **Regularity:**
 - ▶ Anisotropic regularity
 - ▶ Singular behavior in extended variable y .

Spectral Representation of \mathcal{U} (N, Otárola, Salgado (2015))

- **Spectral representation:** $\mathcal{U}(x, y) = \sum_{k=1}^{\infty} u_k \varphi_k(x) \psi_k(y)$ with $u_k = \lambda_k^{-s} f_k$ and $f_k = (f, \varphi_k)$.
- **2-point boundary value problem:** the function ψ_k satisfies

$$\psi_k'' + \frac{\alpha}{y} \psi_k' = \lambda_k \psi_k, \quad \text{in } (0, \infty); \quad \psi_k(0) = 1, \quad \lim_{y \rightarrow \infty} \psi_k(y) = 0,$$

whence for $s \neq \frac{1}{2}$

$$\psi_k(y) = c_s (\sqrt{\lambda_k} y)^s K_s(\sqrt{\lambda_k} y),$$

where $c_s = 2^{1-s}/\Gamma(s)$ and K_s denotes the modified **Bessel function** of the second kind. For $s = \frac{1}{2}$, we have $\psi_k(y) = \exp(-\sqrt{\lambda_k} y)$.

- **Asymptotic behavior:** function ψ_k satisfies as $y \rightarrow 0$

$$\psi_k'(y) \approx y^{-\alpha}, \quad \psi_k''(y) \approx y^{-\alpha-1},$$

and $\psi_k(y) \approx (\sqrt{\lambda_k} y)^{s-\frac{1}{2}} e^{-\sqrt{\lambda_k} y}$ as $y \rightarrow \infty$.

Global Sobolev Regularity (N. Otárola, Salgado (2015))

- **Compatible data:** Let $f \in \mathbb{H}^{1-s}(\Omega)$, which means that f has a vanishing trace for $s < \frac{1}{2}$.
- **Space regularity:**

$$\|\Delta_x \mathcal{U}\|_{L^2(y^\alpha, \mathcal{C})}^2 + \|\partial_y \nabla_x \mathcal{U}\|_{L^2(y^\alpha, \mathcal{C})}^2 = d_s \|f\|_{\mathbb{H}^{1-s}(\Omega)}^2$$

- **Regularity in extended variable y :** If $s \neq \frac{1}{2}$ and $\beta > 2\alpha + 1$ then

$$\|\partial_{yy} \mathcal{U}\|_{L^2(y^\beta, \mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}.$$

If $s = \frac{1}{2}$, then

$$\|\mathcal{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1/2}(\Omega)}.$$

- **Elliptic pick-up regularity:** If Ω convex, then

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta_x w\|_{L^2(\Omega)} \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega).$$

Under this assumption, we further have

$$\|D_x^2 \mathcal{U}\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

Boundary Regularity (Caffarelli, Stinga (2016))

- **Case $s \neq \frac{1}{2}$:** If $\text{dist}(x, \partial\Omega)$ is the distance to $\partial\Omega$, then there exist functions v 'smooth' such that for all $x \in \Omega$

$$u(x) \approx \text{dist}(x, \partial\Omega)^{2s} + v(x) \quad 0 < s < \frac{1}{2}$$

$$u(x) \approx \text{dist}(x, \partial\Omega) + v(x) \quad \frac{1}{2} < s < 1.$$

- **Case $s = \frac{1}{2}$:** This is an exceptional case (Costabel, Dauge (1993))

$$u(x) \approx \text{dist}(x, \partial\Omega) |\log \text{dist}(x, \partial\Omega)| + v(x).$$

Two-Step Algorithm (N, Otárola, Salgado (2015))

- **Domain Truncation** $\mathcal{C}_\gamma := \Omega \times (0, \gamma)$: Let \mathcal{V} solve

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathcal{V}) = 0 & \text{in } \mathcal{C}_\gamma = \Omega \times (0, \gamma), \\ \mathcal{V} = 0 & \text{on } \partial_L \mathcal{C}_\gamma \cup \Omega \times \{\gamma\}, \\ \partial_\nu^\alpha \mathcal{V} = d_s f & \text{on } \Omega \times \{0\}. \end{cases}$$

We get exponential convergence for all $\gamma > 0$,

$$\|\mathcal{U} - \mathcal{V}\|_{\overset{\circ}{H}_L^1(y^\alpha, \mathcal{C}_\gamma)} \lesssim e^{-\sqrt{\lambda_1} \gamma / 4} \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

- **FEM with Anisotropic Mesh:** $\mathcal{T}_\gamma = \{T\}$ is a partition of \mathcal{C}_γ into cells

$$T = K \times I, \quad K \in \mathcal{T}_\Omega, \quad I = (a, b)$$

where $\mathcal{T}_\Omega = \{K\}$ is a conforming and shape regular partition of Ω (simplices or cubes) and neighbor intervals I, I' satisfy the geometric condition

$$\frac{|I|}{|I'|} \simeq 1.$$

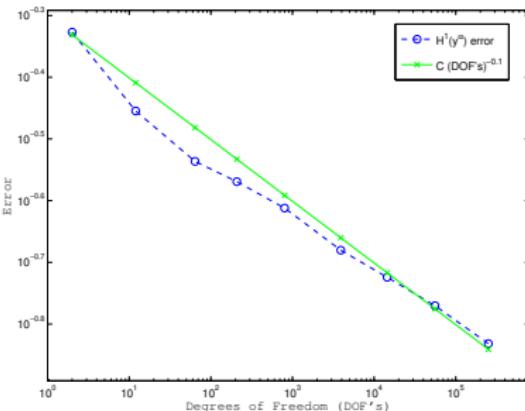
This allows for graded (radical or geometric) meshes towards $y = 0$.

Error Estimates: Quasiuniform Meshes (N, Otárola, Salgado (2015))

- **A priori error estimates for trace:** $U := \mathcal{V}(\cdot, 0)$

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h^{s-\epsilon} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

- **Sharpness:** Is this estimate sharp for quasi-uniform meshes? Consider experiment for $s = 0.2$ and exact solution $\mathcal{U} = \frac{2^{1-s} \pi^s}{\Gamma(s)} \sin(\pi x') y^s K_s(\pi y)$



The energy error behaves like $\text{DOFs}^{-0.1} \approx h^{0.2}$, as predicted! Note that $\text{DOFs} = N_\Omega$ are measured in $\Omega \subset \mathbb{R}^2$.

A Priori Error Estimates: Radical Meshes (N, Otárola, Salgado (2015))

- **Anisotropic interpolation estimates:** exploit tensor product structure.
- **Principle of error equilibration:** We use a **graded mesh** on $(0, \gamma)$

$$y_j = \gamma \left(\frac{j}{M} \right)^\gamma, \quad j = \overline{0, M}, \quad \gamma > 1$$

$\mathcal{U}_{yy} \approx y^{-\alpha-1} \implies$ energy equidistribution for $\gamma > 3/(1-\alpha)$.

- **A priori error estimates.** If $f \in \mathbb{H}^{1-s}(\Omega)$ and $\gamma \approx |\log N_\Omega|$, $h \approx N_\Omega^{-1/d}$

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h |\log h|^s \|u\|_{\mathbb{H}^{1+s}(\Omega)} \approx |\log N_\Omega|^s N_\Omega^{-\frac{1}{d}} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

- **Optimality:**

- ▶ This is **near optimal** in terms of regularity $u \in \mathbb{H}^{1+s}(\Omega)$ and decay rate (almost linear in h);
- ▶ This is **suboptimal** in terms of total degrees of freedom $N = N_\Omega^{1+\frac{1}{d}}$ because of additional dimension that accounts for $N_\Omega^{1/d}$ dofs.
- ▶ Improvements: **sparse tensor FEMs** or **hp -FEMs** (Vexler et al (2017) and Banjai et al (2017)) and **spectral methods** in extended variable (Chen et al (2016), Ainsworth-Glusa (2017)).

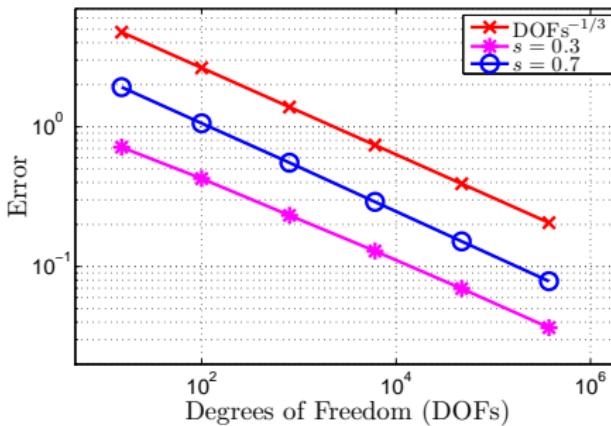
Experimental Rates for a Circle and $s = 0.3, s = 0.7$

- **Domain and exact forcing:** Set $\Omega = B(0, 1) \subset \mathbb{R}^2$ and

$$f = j_{1,1}^{2s} J_1(j_{1,1} r)(A_{1,1} \cos(\theta) + B_{1,1} \sin(\theta)).$$

where J_1 is the 1-st Bessel function of the first kind.

- **Experimental rates of convergence:** With graded meshes we get



- **Optimality:** The experimental convergence rate $-1/3$ is optimal in terms of the total number of DOFs $= N = N_\Omega^{3/2}$.

Diagonalization (w. Banjai, Melenk, Otárola, Salgado, and Schwab (2017))

- **Discretization in y :** Let \mathcal{G}^M be an arbitrary mesh in $(0, \mathcal{Y})$ with $M = \#\mathcal{G}^M$ and let $\mathbb{V}_M^r(\mathcal{C}_{\mathcal{Y}}) = H_0^1(\Omega) \otimes S^r(0, \mathcal{Y}; \mathcal{G}^M)$ be a space of polynomial degree r .
- **Semidiscrete solution:** $\mathcal{U}_M \in \mathbb{V}_M^r(\mathcal{C}_{\mathcal{Y}})$ satisfies

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla \mathcal{U}_M \nabla \phi = d_s \langle f, \text{tr} \phi \rangle \quad \forall \phi \in \mathbb{V}_M^r(\mathcal{C}_{\mathcal{Y}}).$$

- **Discrete eigenvalue problem:** Let $\mathcal{M} = \dim S^r(0, \mathcal{Y}; \mathcal{G}^M)$ and $(\mu_i, v_i)_{i=1}^{\mathcal{M}}$ be (normalized) eigenpairs of

$$\mu \int_{y=0}^{\mathcal{Y}} y^\alpha v'(y) w'(y) dy = \int_{y=0}^{\mathcal{Y}} y^\alpha v(y) w(y) dy \quad \forall w \in S^r(0, \mathcal{Y}; \mathcal{G}^M).$$

- **Representation:** If $\mathcal{U}_M(x', y) = \sum_{j=1}^{\mathcal{M}} U_j(x') v_j(y)$ with $U_j \in H_0^1(\Omega)$, then

$$a_{\mu_i, \Omega}(U_i, V) = d_s v_i(0) \langle f, V \rangle \quad \forall V \in H_0^1(\Omega) \quad \Rightarrow \quad \text{Parallelization!}$$

where $a_{\mu_i, \Omega}$ is the singularly perturbed bilinear form

$$a_{\mu_i, \Omega}(U, V) := \mu_i \int_{\Omega} \nabla U \nabla V dx' + \int_{\Omega} UV dx$$

Optimal FEMs (w. Banjai, Melenk, Otárola, Salgado, Schwab (2017))

- **Complexity of tensor product:** quantity $N = N_\Omega^{1+\frac{1}{2}}$ is **suboptimal**.
- **Sparse grid space:** Let

$$\mathbb{V}_L^{1,1}(\mathcal{C}_{\mathcal{Y}}) = \sum_{\ell, \ell' \geq 0, \ell + \ell' \leq L} S_0^1(\mathcal{T}_\Omega^\ell) \otimes S^1(0, \mathcal{Y}; \mathcal{G}_\eta^{2\ell'}),$$

where \mathcal{T}_Ω^ℓ and $\mathcal{G}_\eta^{2\ell'}$ are nested meshes of levels ℓ and ℓ' graded towards corners \mathbf{c} of Ω and $y = 0$ (grading dictated by $\eta > 1$), respectively. Then

$$\dim \mathbb{V}_L^{1,1}(\mathcal{C}_{\mathcal{Y}}) \lesssim N_\Omega \log \log N_\Omega.$$

- **Error estimate:** Let $1 < \nu < 1 + s$, $\eta(\nu - 1) \geq 1$, and $\mathcal{Y} \approx |\log h_L|$. If $f \in \mathbb{H}^{\nu-s}(\Omega)$, then $\mathcal{U}_L \in \mathbb{V}_L^{1,1}(\mathcal{C}_{\mathcal{Y}})$ satisfies

$$\|\mathcal{U} - \mathcal{U}_L\|_{L^2(y^\alpha, \mathcal{C})} \lesssim h_L |\log h_L| \|f\|_{\mathbb{H}^{\nu-s}(\Omega)}.$$

- ***hp*-FEM in y :** geometric mesh with linear growth of polynomial degree yield linear error estimates for non-convex domains. This exploits analyticity in y .
- **Complexity:** sparse grids and *hp*-FEM in y are **quasi-optimal** in terms of N_Ω .

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Outline

Motivation

The Integral Laplacian

The Spectral Laplacian

Dunford-Taylor Approach

Extensions

Open Problems

Integral Representation of Spectral Laplacian (Bonito and Pasciak (2015))

- **Balakrishnan Formula:** deforming the contour of a Dunford integral yields

$$u = (-\Delta)^{-s} f = \underbrace{\frac{\sin(\pi s)}{\pi}}_{=C(s)} \int_0^\infty \mu^{-s} (\mu I - \Delta)^{-1} f d\mu.$$

- **Sanity Check:** If $\psi \in H_0^1(\Omega)$ is an eigenfunction of $(-\Delta)$ with associated eigenvalue $\lambda > 0$ then

$$(-\Delta)^{-s} \psi = C(s) \psi \int_0^\infty \frac{\mu^{-s}}{\mu + \lambda} d\mu \stackrel{\mu = \lambda t}{=} \lambda^{-s} C(s) \psi \underbrace{\int_0^\infty \frac{t^{-s}}{t + 1} dt}_{=C(s)^{-1}} = \lambda^{-s} \psi.$$

- **Three-step algorithm:**

- ▶ Step 1: use quadrature for the μ variable;
- ▶ Step 2: use standard finite element methods on the **same mesh** to approximate

$$u_\mu \in H_0^1(\Omega) : \quad \mu u_\mu - \Delta u_\mu = f \quad \text{in } \Omega,$$

or equivalently $u_\mu = (\mu I - \Delta)^{-1} f$.

- ▶ Step 3: gather all contributions.

Step 1: SINC Quadrature for the μ Variable

- **Change of variable:** let $\mu = e^y$ to get

$$u = (-\Delta)^{-s} f = \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{\infty} e^{(1-s)y} (e^y I - \Delta)^{-1} f dy.$$

- **Quadrature:** Given $N \in \mathbb{N}$, define $k = 1/\sqrt{N}$, $y_j = jk$ and the quadrature approximation

$$U^N = \underbrace{\frac{\sin(\pi s)k}{\pi}}_{=C(s,k)} \sum_{j=-N}^N e^{(1-s)y_j} (e^{y_j} I - \Delta)^{-1} f.$$

- **Exponential convergence** (Bonito, Pasciak (2015)): Let $s \in [0, 1)$ and $r \in [0, 1]$. If $f \in \mathbb{H}^r(\Omega)$, then

$$\|u - U^N\|_{\mathbb{H}^r(\Omega)} \leq C e^{-c\sqrt{N}} \|f\|_{\mathbb{H}^r(\Omega)}.$$

In practice $N = 20$. This uses decay when $|z| \rightarrow \infty$ and holomorphic properties of integrand $z^{-s}(zI - \Delta)^{-1}$.

Step 2: Finite Element Method and Parallelization

- **Fully discrete solution:** Let $U = U_{\mathcal{T}}^N \in \mathbb{U}(\mathcal{T})$ satisfy

$$U = C(s, k) \sum_{j=-N}^N e^{(1-s)y_j} \underbrace{(e^{y_j} I - \Delta_{\mathcal{T}})^{-1} \Pi_{\mathcal{T}} f}_{=U_j},$$

where $-\Delta_{\mathcal{T}}$ is the discrete Laplacian and $\Pi_{\mathcal{T}}$ the L^2 -projection onto the discrete space $\mathbb{U}(\mathcal{T})$.

- **Parallelization:** Each $U_j \in \mathbb{U}(\mathcal{T})$ solves $(e^{y_j} I - \Delta_{\mathcal{T}})U_j = \Pi_{\mathcal{T}} f$, i.e.

$$\int_{\Omega} e^{y_j} U_j W + \nabla U_j \nabla W = \int_{\Omega} f W \quad \forall W \in \mathbb{U}(\mathcal{T}).$$

A Priori Error Analysis: Comparison with the Extension Approach

- **Convex domains:** $(-\Delta)^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ full elliptic pick-up regularity.
- **Comparison 1:** The discrete Balakrishnan scheme gives the error bound

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \leq C h^{2-s} \|f\|_{\mathbb{H}^{2-2s}(\Omega)}.$$

- ▶ This estimate is of optimal order $2 - s > 1$ and regularity $f \in \mathbb{H}^{2-2s}(\Omega)$;
- ▶ This formally corresponds to $u \in H^2(\Omega)$ that is not generic: $u \in H^{\frac{1}{2}+s-\epsilon}(\Omega)$;
- ▶ In contrast, the Extension Approach cannot deliver orders larger than 1.
- **Comparison 2:** Extension approach requires $f \in \mathbb{H}^{1-s}(\Omega)$ to deliver order 1 accuracy. What is the regularity of f for order 1 with Dunford-Taylor?
- $f \in \mathbb{H}^{1-s}(\Omega).$
- **Comparison 3:** Both approaches lead to parallel algorithms.

Definition of Integral Laplacian (Bonito, Lei, Pasciak (2017))

- **Fourier definition:**

$$\begin{aligned} \llbracket u, w \rrbracket &= C \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(x'))(w(x) - w(x'))}{|x - x'|^{d+2s}} dx dx' \\ &= \int_{\mathbb{R}^d} |\xi|^s \mathcal{F}(u)(\xi)^s \overline{\mathcal{F}(w)} d\xi = \int_{\mathbb{R}^d} \mathcal{F}((- \Delta)^s u)(\xi) \overline{\mathcal{F}(w(\xi))} d\xi = (f, w). \end{aligned}$$

- **Equivalent representation:**

$$\llbracket u, w \rrbracket = \frac{2 \sin(s\pi)}{\pi} \int_0^\infty \mu^{1-2s} \int_{\mathbb{R}^d} (- \Delta(I - \mu^2 \Delta)^{-1} u) w dx d\mu.$$

- **Idea of Proof:**

- ▶ Parseval's theorem:

$$\int_{\mathbb{R}^d} (- \Delta(I - \mu^2 \Delta)^{-1} u) w dx = \int_{\mathbb{R}^d} \frac{|\xi|^2}{1 + \mu^2 |\xi|^2} \mathcal{F}(u)(\xi) \overline{\mathcal{F}(w)(\xi)} d\xi.$$

- ▶ Change of variables: $t = \mu |\xi|$ yields

$$\int_0^\infty \frac{t^{1-2s}}{1+t^2} dt = \frac{\pi}{2 \sin(\pi s)}.$$

Variational Formulation

- **Auxiliary problem:** given $\psi \in L^2(\mathbb{R}^d)$ let $v(\psi, \mu) = v(\mu) \in H^1(\mathbb{R}^d)$ satisfy

$$v - \mu^2 \Delta v = -\psi \quad \Rightarrow \quad v = -(I - \mu^2 \Delta)^{-1} \psi.$$

which corresponds to the weak formulation

$$\int_{\mathbb{R}^d} v(\mu) \phi + \mu^2 \int_{\mathbb{R}^d} \nabla v(\mu) \cdot \nabla \phi = - \int_{\mathbb{R}^d} \psi \phi \quad \forall \phi \in H^1(\mathbb{R}^d),$$

Note that the support of $v(\psi, \mu)$ is all of \mathbb{R}^d regardless of the support of ψ .

- **Equivalent expression of $[\![u, w]\!]$:** Using that $\Delta v(u) = \mu^{-2}(v(u) + u)$ gives

$$[\![u, w]\!] = \frac{2 \sin(s\pi)}{\pi} \int_0^\infty \mu^{-1-2s} \left(\int_\Omega (u + v(u, \mu)) w \, dx \right) d\mu \quad \forall u, w \in \mathbb{H}^s(\Omega).$$

- **Variational problem:** given $f \in \mathbb{H}^{-s}(\Omega)$ find $u \in \mathbb{H}^s(\Omega)$ such that

$$[\![u, w]\!] = (f, w) \quad \forall w \in \mathbb{H}^s(\Omega).$$

Three-Step Algorithm

- **Sinc quadrature:** the change of variables $\mu = e^{-\frac{y}{2}}$ implies

$$[\![u, w]\!] = \frac{\sin(s\pi)}{\pi} \int_{-\infty}^{\infty} e^{sy} \left(\int_{\Omega} (u + v(u, \mu(y))) w \, dx \right) dy$$

whence uniform spacing $k \approx 1/N$ and $y_j = jk$ yields

$$[\![u, w]\!]^N = \frac{\sin(s\pi)}{\pi} k \sum_{j=-N}^N e^{sy_j} \left(\int_{\Omega} (u + v(u, \mu(y_j))) w \, dx \right) dy$$

- **Domain Truncation:** since $\text{supp } v(u, \mu) = \mathbb{R}^d$ we solve on a ball $B^M(\mu)$ containing Ω of radius dictated by M and μ .
- **Finite element approximation:** if partitions of Ω and $B^M(\mu) \setminus \Omega$ are compatible, then the fully discrete bilinear form reads

$$[\![U, W]\!]_{\mathcal{T}}^{N,M} = \frac{\sin(s\pi)}{\pi} k \sum_{j=-N}^N e^{sy_j} \int_{\Omega} (U + V^M(U, \mu(y_j))) W \, dx,$$

where $U = U_{\mathcal{T}}^{N,M}$ is the FE approximation of u in Ω and $V^M(U, \mu(y_j))$ is the FE approximation of $v(U, \mu(y_j))$ in $B^M(\mu)$.

Fully Discrete Scheme

- **Finite element solution:** find $U = U_{\mathcal{T}}^{N,M} \in \mathbb{U}(\mathcal{T})$ such that

$$[U, W]_{\mathcal{T}}^{M,N} = \int_{\Omega} fW \quad \forall W \in \mathbb{U}(\mathcal{T}).$$

- **A priori error estimate:** Let $\beta \in (s, 3/2)$. Then

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim \left(e^{-c\sqrt{N}} + e^{-cM} + |\log h| h^{\beta-s} \right) \|u\|_{\mathbb{H}^\beta(\Omega)}.$$

- **Rate of convergence:** Take $\beta = s + \frac{1}{2} - \epsilon$, that is consistent with the regularity of $u \in \mathbb{H}^{\frac{1}{2}+s-\epsilon}(\Omega)$, and $M \approx |\log h|$, $N \approx |\log h|^2$, to obtain

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h^{\frac{1}{2}-\epsilon} \|u\|_{\mathbb{H}^{\frac{1}{2}+s-\epsilon}(\Omega)}.$$

- **Comparison with integral method:**

- ▶ Similar convergence rate for **quasi-uniform \mathcal{T}**
- ▶ Effect of locally refined meshes towards $\partial\Omega$ remains open: improved rate?

Outline

Motivation

The Integral Laplacian

The Spectral Laplacian

Dunford-Taylor Approach

Extensions

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Extensions

- **Sparse tensor FEMs and hp -FEMs:** Meidner, Pfefferer, Schürholz, Vexler (2017); Banjai, Melenk, N, Otárola, Salgado, Schwab (2017).
- **Time-dependent problems (Caputo):** N, Otárola, Salgado (2016); Bonito, Lei, Pasciak (2017); Acosta, Bersetche, and Borthagaray (2017).
- **Multilevel solvers:** Chen, N, Otárola, Salgado (2016); Ainsworth, Glusa (2017); Baerland, Kuchta, Mardal (2018).
- **Obstacle problems:** Schwab, Matacle, Nitsche (2005); N. Otarola, Salgado (2015); Borthagaray, N, Salgado (2018); Bonito, Lei, Salgado (2018); Burkova, Gunzburger (2018).
- **A posteriori error analysis:** N, Von Petersdorff, Zhang (2010); Chen, N, Otárola, Salgado (2014); Ainsworth, Glusa (2017).
- **Spectral methods:** Chen, Shen, Wang (2016); Ainsworth, Glusa (2016); Antil, Bartels (2017); Karniadakis et al (2014-17).
- **Control:** Antil, Otárola, Salgado (2015-2017).
- **Eigenvalue problems** (Borthagaray, Del Pezzo, Martínez (2017))
- **Nonhomogeneous Dirichlet conditions:** Acosta, Borthagaray, Heuer (2017).

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- **Computations in 3d:** implementation of fractional Laplacian; regularity and numerical analysis are valid for $d > 2$.
- **High-order methods:** hp -FEM with suitable mesh refinement near boundary might yield exponential convergence rates.
- **Efficiency:** Compression techniques and fast multilevel solvers (Ainsworth, Glusa (2017); Karkulik, Melenk (2018)).
- **Quadrature:** Error analysis of effect of quadrature close to singularities of kernel (Sauter and Schwab (2011)).
- **A posteriori error analysis:** implementation for $d > 1$ of residual-type estimators for integral Laplacian (Ainsworth, Glusa (2017)); alternative approaches; ideal estimator for spectral Laplacian (Chen, N, Otárola, Salgado (2015)); Dunford-Taylor approach.
- **Nonlinear problems:** obstacle (parabolic), fractional minimal surfaces, fractional phase transitions (Ainsworth, Mao (2017); Antil, Bartels (2017)), fractional fully-nonlinear problems.

Survey Paper

A. Bonito, J.P. Borthagaray, R.H. Nochetto, E. Otárola, A.J. Salgado,
Numerical Methods for Fractional Diffusion, Computing and Visualization in
Science (2018), 1–28; arXiv:1707.01566.